数据包络分析软件 MaxDEA X

Data Envelopment Analysis Software MaxDEA X

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第 9 章 新型两阶段网络 DEA 与广义网络 DEA

数据包络分析(DEA)作为评估决策单元(DMU)相对效率的非参数核心工具，其传统模型是将 DMU 视为内部结构不可见的“黑箱”，仅关注最终投入与产出的转化关系。然而，真实生产系统内部往往包含复杂的子流程、多阶段结构以及相互流转的中间流(Intermediates)。网络 DEA(Network DEA)的提出，正是为了打开这一“黑箱”，分析系统内部各节点(阶段)[i]的运行效率及其对系统整体的影响。在网络 DEA 的技术逻辑中，各个生产阶段在技术上具备明确的独立性，即每个节点各自构建其生产可能集(PPS)。这种技术独立性使得将一个 DMU 在阶段 1 的技术与另一个 DMU 在阶段 2 的技术进行线性组合在技术上是可行且合规的。

尽管两阶段及广义网络 DEA(涵盖多节点、串并联及任意拓扑结构)极大地扩展了传统 DEA 的应用范围，但在理论与算法层面长期面临一个痛点：**系统效率分解的不唯一性，并且分解得出的节点效率值往往并非该节点在其独立 PPS 下的效率值**。在传统网络 DEA 的乘数模型求解过程中，由于可能存在多重最优解，系统总效率与各子节点效率之间的代数映射关系并不稳定。这种不稳定的分解为网络 DEA 模型节点效率值的进一步分析（例如节点的 TFP 指数分析）设置了障碍。为解决这一问题，既往研究多依赖人为设定二次目标、引入主观偏好权重或构建主从博弈模型来强行“锁定”分解结果。

MaxDEA 团队开发的新型两阶段网络 DEA 模型从根本上解决了这一难题，其系统效率与两个阶段的效率之间天然具备唯一性等式关系：**系统整体效率值等于两阶段唯一效率值之乘积。**在此基础上，将其推广到广义网络 DEA 和基于全维有效面(FDEF)的广义网络 DEA。

若您在学术研究或论文中引用本文，请参考以下引文格式：

英文格式： MaxDEA Team. (2026-10-26). *Novel Two-Stage Network DEA and Generalized Network DEA*. MaxDEA. https://maxdea.com/2026/202609.pdf.

中文格式： MaxDEA 团队. 新型两阶段网络 DEA 和广义网络 DEA [EB/OL]. (2026-10-26) [YYYY-MM-DD]. https://maxdea.com/2026/202609.pdf. [YYYY-MM-DD]为引用日期。

[i] 本文在两阶段网络 DEA 中，使用“阶段(Stage)”，在广义网络 DEA 中，使用“节点(Node)”，来表示网络的内部结构。另外，对于权重向量 v 和 u，会混用“权重”和“（影子）价格”。

符号及数据说明

公式采用矩阵和向量表达式。为方便读者复制公式，尽量采用**纯文本格式**。

矩阵：X、Y、Z 是投入、产出和中间流矩阵。在乘数模型中矩阵行数等于样本 DMU 数量，在包络模型中矩阵列数等于样本 DMU 数量。

X、Y、Z > 0。

列向量：x_o、y_o、z_o表示当前被评价 DMU 的投入、产出和中间流向量(使用下标 $_o$ 在视觉上更为醒目)。λ 为包络模型中的 DMU 线性组合系数。

行向量：v、u、w 是乘数模型中投入、产出和中间流的权重。

v、u、w、λ，默认为非负变量 (≥ 0)。

用粗体 **1** 表示所有元素均为 1 的行向量或列向量。

$u_o \in R$ 是标量变量，用于 VRS 乘数模型，表示规模报酬常数项(截距项)。

上标：$^{(1)}$ $^{(2)}$ $^{(t)}$ 表示阶段 1、阶段 2、节点 t；$^{(in)}$ $^{(out)}$ 表示中间流在节点 t 中的角色是投入还是产出。

9.1 两阶段网络 DEA (规模报酬不变 CRS)

9.1.1 两阶段网络 DEA 模型

Kao & Hwang (2008)和 Liang, Cook, & Zhu (2008)提出了将系统整体效率定义为两个节点效率乘积形式的两阶段网络 DEA 模型，开创了两阶段网络 DEA 模型的乘积效率分解框架(Multiplicative Decomposition)。

阶段 1 的效率

$E^{(1)}* = \max (w \cdot z_o / v \cdot x_o)$

$s.t. \quad (w \cdot Z) / (v \cdot X) \le 1$ (9.1)

阶段 2 的效率

$E^{(2)}* = \max (u \cdot y_o / w \cdot z_o)$

$s.t. \ (u \cdot Y) / (w \cdot Z) \le 1$ (9.2)

系统整体效率

$E* = E^{(1)} \times E^{(2)} = \max (u \cdot y_o / w \cdot z_o) \times (w \cdot z_o / v \cdot x_o) = \max (u \cdot y_o / v \cdot x_o)$

$s.t. \quad (w \cdot Z) / (v \cdot X) \le 1$

$(u \cdot Y) / (w \cdot Z) \le 1$ (9.3)

利用 Charnes-Cooper 变换，得到如下线性规划(LP)模型（投入导向）：

$\max u \cdot y_o$

$s.t. \ v \cdot X - w \cdot Z \ge 0$

$w \cdot Z - u \cdot Y \ge 0$

$v \cdot x_o = 1$ (9.4)

对偶包络模型

$\min \theta$

$s.t. \ X \cdot \lambda^{(1)} \le x_o \cdot \theta$

$Y \cdot \lambda^{(2)} \ge y_o$

$Z \cdot \lambda^{(1)} \ge Z \cdot \lambda^{(2)}$

$\theta \in R$ (9.5)

两阶段网络 DEA 的系统整体效率可分解为两部分的乘积：

$E^* = (u \cdot y_o) / (v \cdot x_o) = (u \cdot y_o) / (w \cdot z_o) \cdot (w \cdot z_o) / (v \cdot x_o) = E^{(1)} \times E^{(2)} \le E^{(1)*} \times E^{(2)*}$

上述两阶段网络 DEA 框架通过打开系统“黑箱”，成功刻画了系统内部中间产品的流转过程。在其理论架构下，模型表现出以下特征和缺陷：

模型特征 1：中间流价格相等的刚性协同(Rigid Synergy)

乘数模型严格要求阶段 1 中间流（作为产出）的权重 $w^{(out)}$与阶段 2 中间流（作为投入）的权重 $w^{(in)}$必须“一对一”绝对相等。

模型特征 2：技术改进路径的混杂性

刚性协同机制映射到对偶包络模型中，表现为径向与非径向的混合——模型仅对初始投入 x（在产出导向模型中是最终产出 y）施加径向等比例缩放约束(θ)，而中间流 z 则具有完全的自由度。这种混杂结构削弱了模型的经济学含义，并且导致投影偏离了经典的技术改进路径。

缺陷 1：系统效率分解的不唯一性

在此框架下，尽管系统整体效率唯一确定，但由于乘数模型往往存在多重最优解，得出的阶段效率分解值并不具备唯一性。

$E^* = E^{(1)} \times E^{(2)}$，$E^{(1)}$和 $E^{(2)}$不具备唯一性。

缺陷 2：分解得出的阶段效率值可能偏离各自的 PPS

在刚性协同机制下分解得出的阶段效率，不一定是各阶段在各自 PPS 下的真实技术效率。

$E^{(1)} \le E^{(1)*}$，$E^{(2)} \le E^{(2)*}$。

9.1.2 基于中间流虚拟价值守恒的两阶段网络 DEA 模型

针对上述框架存在的痛点，提出如下中间流总价值相等的柔性协同(Flexible Synergy)的两阶段网络 DEA 模型：

$$E^* = \max (u \cdot y_o) / (v \cdot x_o)$$

$$\text{s.t.} \quad (w^{(1)} \cdot Z) / (v \cdot X) \le 1$$

$$(u \cdot Y) / (w^{(2)} \cdot Z) \le 1$$

$$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o} \qquad (9.6)$$

利用 Charnes-Cooper 变换，得到如下线性规划(LP)模型（投入导向）：

$$E^* = \max u \cdot y_o$$
$$\text{s.t. } v \cdot X - w^{(1)} \cdot Z \ge 0$$
$$w^{(2)} \cdot Z - u \cdot Y \ge 0$$
$$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o}$$

$$v \cdot x_o = 1 \qquad (9.7)$$

对偶包络模型

$$E^* = \min \theta$$

$$s.t.\ X \cdot \lambda^{(1)} \le x_o \cdot \theta$$

$$\mathbf{Z \cdot \lambda^{(1)} \ge z_o \cdot \tau}$$

$$\mathbf{Z \cdot \lambda^{(2)} \le z_o \cdot \tau}$$

$$Y \cdot \lambda^{(2)} \ge y_o$$

$$\theta, \tau \in R \qquad (9.8)$$

或者利用 Charnes-Cooper 变换，得到如下线性规划(LP)模型（产出导向）：

$$1/E^* = \min v \cdot x_o$$
$$s.t.\ v \cdot X - w^{(1)} \cdot Z \ge 0$$
$$w^{(2)} \cdot Z - u \cdot Y \ge 0$$
$$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o}$$
$$u \cdot y_o = 1 \qquad (9.9)$$

对偶包络模型

$$1/E^* = \max \phi$$

$$s.t.\ X \cdot \lambda^{(1)} \le x_o$$

$$\mathbf{Z \cdot \lambda^{(1)} \ge z_o \cdot \tau}$$

$$\mathbf{Z \cdot \lambda^{(2)} \le z_o \cdot \tau}$$

$$Y \cdot \lambda^{(2)} \ge y_o \cdot \phi$$

$$\phi, \tau \in R \qquad (9.10)$$

基于中间流虚拟价值守恒的两阶段网络 DEA 模型具备以下特性：

1) 中间流柔性协同

放弃每个中间流价格必须相等的强外生约束，转而要求被评价 DMU 的所有中间流在系统内部流转时满足“总价值相等”，符合中间流内部结算不产生外部性的流转特征，使中间流权重（影子价格）可以在系统内部依据经典 DEA 的最优化原则确定。

$$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o}$$

2) 纯粹的经典技术改进路径

完全采用经典的径向等比例技术改进路径，完整保留经典 DEA 模型的经济学意义。

$$Z \cdot \lambda^{(1)} \geq z_o \cdot \tau$$

$$Z \cdot \lambda^{(2)} \leq z_o \cdot \tau$$

3) 系统效率分解具有唯一性，并且分解出的阶段效率值是各自 PPS 下的技术效率。

9.1.3 两阶段网络 DEA 模型(CRS)的效率分解

两阶段网络 DEA 模型(9.6)，被评价 DMU 的整体网络效率 **E*** 等于阶段 1 模型(9.1)和阶段 2 模型(9.2)求解出的最优阶段效率 $E^{(1)}*$ 与 $E^{(2)}*$ 之积（**证明附后**）：

$$E* = E^{(1)}* \times E^{(2)}*$$

9.1.4 中间流柔性协同机制

在网络 DEA 模型中，阶段独立确定的影子价格仅在阶段内部有效，其影子价格可以等效地整体放大或缩小。由于两个阶段的生产技术各自独立，当对系统整体效率进行评价时，需要一套协同机制来反映中间流在网络结构中的桥梁角色和中间流的流转规则。

柔性协同(Flexible Synergy)放弃了刚性协同(Rigid Synergy)的严苛约束（中间流价格一对一相等，$w^{(1)} = w^{(2)}$），放宽为“中间流在系统内部的生产端和消耗端总价值相等”($w^{(1)} \cdot z_o = w^{(2)} \cdot z_o$)。正是这项核心改变，实现了系统效率唯一性分解并使分解出的阶段效率与阶段独立技术效率保持一致。

从影子价格和生产函数理论视角剖析

在对偶规划与线性规划理论中，乘数可直接解释为各类投入、产出及中间流的内部影子价格(Shadow Prices)或边际估值。

刚性协同强制要求 $w^{(in)} = w^{(out)}$，为中间产品设置了一个跨阶段统一的“外生市场价格”。然而，阶段 1 与阶段 2 属于不同的生产技术，中间流在两个阶段的边际替代率(MRS)并不相同。强行要求中间流在两个阶段的影子价格一一相等，相当于用统一的线性切面去强行拟合两个独立的凸集前沿，必然导致拟合的前沿与两个阶段的前沿出现偏离，阶段效率被刚性协同约束所扭曲。

柔性协同仅约束中间流在系统内部转移时保持价值守恒，这赋予了 $w^{(1)}$和 $w^{(2)}$独立的自由度，使其能够分别精确匹配两阶段各自最优前沿面的法向量（即各自独立的局部最佳影子价格），从而使网络系统评估的阶段技术效率与该阶段独立评估的技术效率相一致。

从加权虚拟投入产出理论视角剖析

经典 DEA 的核心思想在于将多投入多产出聚合为标量型的虚拟投入(Virtual Input)与虚拟产出(Virtual Output)，并在虚拟产出/虚拟投入比值的约束下(<=1)，给被评价 DMU 完全的自由度来寻求最优权重。

中间流具有双重身份，既是阶段 1 的虚拟产出，同时又是阶段 2 的虚拟投入。刚性协同要求单个权重向量同时匹配两套投入产出结构，这限制了阶段各自的寻优自由度，导致模型无法实现阶

段的最优解。柔性协同解除了这一过度约束，仅施加中间产品虚拟聚合值相等约束。这既保证了阶段 1 向阶段 2 传递的虚拟价值在交界处既不凭空产生，也不无故消灭，完美契合中间产品仅存在于网络系统内部，不产生外部性的闭环特征，同时又保留了被评价 DMU 完全的自由度来寻求各阶段的最优权重，使各阶段在网络 DEA 架构下仍然能够实现各自的最优解。

表 9-1 刚性协同与柔性协同对中间流的约束比较

	刚性协同	柔性协同
乘数模型	刚性 中间流在两个阶段价格一对一相等 $w^{(1)} = w^{(2)}$	柔性 中间流在两个阶段总价值相等 $w^{(1)} \cdot z_o = w^{(2)} \cdot z_o$
包络模型	自由缩放 仅要求遵守流转逻辑 $z_i \cdot \lambda^{(2)} \le \hat{z}_i \le z_i \cdot \lambda^{(1)}, \forall_i$ 每项中间流独立自由缩放	径向缩放 遵守流转逻辑和中间流径向变动 $z_i \cdot \lambda^{(2)} \le z_{io} \cdot \tau \le z_i \cdot \lambda^{(1)}, \forall_i$ 或者表示为 $z_i \cdot \lambda^{(2)} \le \hat{z}_i \le z_i \cdot \lambda^{(1)}, \forall_i$ $\hat{z}_i = z_{io} \cdot \tau$ τ 为中间流缩放标量

* 当网络中只有一项中间流时，两种协同方式等价，$w^{(1)} \cdot z_o = w^{(2)} \cdot z_o \rightarrow w^{(1)} = w^{(2)}$。

9.1.5 为什么两阶段网络模型的效率分解要使用乘积形式

两阶段系统效率分解的乘性结构避免了高效率阶段对低效率阶段的过度替代补偿。“两个阶段都不能落后”是两阶段网络系统的必然要求，效率乘积形式体现了典型的串联系统性质，任意子阶段的低效率（$E^{(t)*} < 1$）都会以乘法形式拉低系统整体效率。

举例说明：两家公司 A 和 B 用相同原料生产相同产品，A 公司在第一阶段用 100 件原料生产了 100 件中间产品，B 公司在第一阶段用 100 件原料生产了 10 件中间产品，与 A 相比，B 在第一阶段的效率是 0.1；A 公司在第二阶段利用 100 件中间产品生产了 100 件最终产品；B 公司在第二阶段利用 10 件中间产品生产了 9 件最终产品，与 A 相比，B 在第二阶段的效率是 0.9。乘积形式正确反映了这种串联连锁效应，B 的整体效率是 0.09，而不是算术平均形式得出的 0.5。

9.2 两阶段网络模型（规模报酬可变 VRS）

9.2.1 模型定义

1) 投入导向模型（截距归入产出端）：

阶段 1 模型：

$E^{(1)}* = \max (w^{(1)} \cdot z_o - u_o^{(1)}) / (v \cdot x_o)$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$ (9.11)

阶段 2 模型：

$E^{(2)}* = \max (u \cdot y_o - u_o^{(2)}) / (w^{(2)} \cdot z_o)$

$s.t.\ w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$ (9.12)

两阶段网络模型：

$E* = \max (w^{(1)} \cdot z_o - u_o^{(1)}) / (v \cdot x_o) \cdot (u \cdot y_o - u_o^{(2)}) / (w^{(2)} \cdot z_o) = \max (u \cdot y_o - u_o^{(2)}) / (v \cdot x_o)$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$

$w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$

$(w^{(1)} \cdot z_o - u_o^{(1)}) = w^{(2)} \cdot z_o$ (9.13)

利用 Charnes-Cooper 变换，转换为线性规划。

$E* = \max (u \cdot y_o - u_o^{(2)})$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$

$w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$

$v \cdot x_o = 1$

$w^{(1)} \cdot z_o - u_o^{(1)} = w^{(2)} \cdot z_o$ (9.14)

对偶包络模型：

$E* = \min \theta$

$s.t.\ X \cdot \lambda^{(1)} \leq x_o \cdot \theta$

$Z \cdot \lambda^{(1)} \geq z_o \cdot \tau$

$Z \cdot \lambda^{(2)} \leq z_o \cdot \tau$

$Y \cdot \lambda^{(2)} \geq y_o$

$\mathbf{1} \cdot \lambda^{(1)} = \tau$

$\mathbf{1} \cdot \lambda^{(2)} = 1$

$\theta, \tau \in R$ (9.15)

2) 产出导向模型（截距归入投入端）：

阶段 1 模型：

$$E^{(1)}* = \max (w^{(1)} \cdot z_o - u_o^{(1)}) / (v \cdot x_o)$$
$$\text{s.t. } v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0 \qquad (9.16)$$

阶段 2 模型：

$$E^{(2)}* = \max (u \cdot y_o - u_o^{(2)}) / (w^{(2)} \cdot z_o)$$

$$\text{s.t. } w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0 \qquad (9.17)$$

两阶段网络模型：

$$E* = \max (w^{(1)} \cdot z_o) / (v \cdot x_o + u_o^{(1)}) \cdot (u \cdot y_o) / (w^{(2)} \cdot z_o + u_o^{(2)}) = \max (u \cdot y_o) / (v \cdot x_o + u_o^{(1)})$$

$$\begin{aligned} &\text{s.t. } v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0 \\ &\quad w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0 \\ &\quad w^{(1)} \cdot z_o = w^{(2)} \cdot z_o + u_o^{(2)} \qquad (9.18) \end{aligned}$$

利用 Charnes-Cooper 变换，转换为线性规划。

$$1/E* = \min (v \cdot x_o + u_o^{(1)})$$

$$\begin{aligned} &\text{s.t. } v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0 \\ &\quad w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0 \\ &\quad u \cdot y_o = 1 \\ &\quad w^{(1)} \cdot z_o = w^{(2)} \cdot z_o + u_o^{(2)} \qquad (9.19) \end{aligned}$$

对偶包络模型：

$$1/E* = \max \phi$$

$$\begin{aligned} &\text{s.t. } X \cdot \lambda^{(1)} \leq x_o \\ &\quad Z \cdot \lambda^{(1)} \geq z_o \cdot \tau \\ &\quad Z \cdot \lambda^{(2)} \leq z_o \cdot \tau \\ &\quad Y \cdot \lambda^{(2)} \geq \phi \cdot y_o \\ &\quad \mathbf{1} \cdot \lambda^{(1)} = 1 \\ &\quad \mathbf{1} \cdot \lambda^{(2)} = \tau \\ &\quad \phi, \tau \in R \qquad (9.20) \end{aligned}$$

9.2.2 效率分解

两阶段网络 DEA 模型(9.13)，被评价 DMU 的整体网络效率 $\mathbf{E^*}$ 等于阶段 1 模型(9.11)和阶段 2 模型(9.12)求解出的最优阶段效率 $\mathbf{E^{(1)}*}$ 与 $\mathbf{E^{(2)}*}$ 之积（与 CRS 模型原理相同，参见附录“CRS 两阶段网络 DEA 效率乘积分解证明”）：

$$\mathbf{E^* = E^{(1)}* \times E^{(2)}*}$$

9.3 广义网络 DEA 模型

9.3.1 模型定义

$E^* = \max \Sigma((u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) / (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)})) / N$

$s.t.\ v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0, \forall t$

$\Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + (\eta_3^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) = 0$ (9.21)

N 为网络的节点数量

η 用于表示 VRS 模型中截距项的归属：$\eta_1^{(t)} + \eta_2^{(t)} + \eta_3^{(t)} + \eta_4^{(t)} = 1, \eta \in \{0, 1\}, \forall t$

如果节点(t)没有投入 $x^{(t)}$：$\eta_1^{(t)} = 0$

如果节点(t)没有产出 $y^{(t)}$：$\eta_2^{(t)} = 0$

如果节点(t)没有中间流 $z^{(in)(t)}$：$\eta_3^{(t)} = 0$

如果节点(t)没有中间流 $z^{(out)(t)}$：$\eta_4^{(t)} = 0$

投入导向模型中截距项归入产出端：$\eta_2^{(t)} + \eta_4^{(t)} = 1$，如果节点(t)有产出 $y^{(t)}$，$\eta_2^{(t)} = 1$

产出导向模型中截距项归入投入端：$\eta_1^{(t)} + \eta_3^{(t)} = 1$，如果节点(t)有投入 $x^{(t)}$，$\eta_1^{(t)} = 1$

CRS 模型：η 均为 0

9.3.2 模型求解

先求解节点独立模型，获得每个节点的最优效率值 $E^{(t)*}$：

$E^{(t)*} = \max (u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) / (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)})$

$s.t.\ v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0$ (9.22)

广义网络模型的目标函数为分式相加形式，属于非线性规划，并且需要允许节点权重进行等比例放大，无法通过 Charnes-Cooper 变换实现线性化，需要采用 Dinkelbach 参数化方法求解。

参数化模型（投入导向，$\theta^{(t)}$是模型参数）：

$D^* = \max \Sigma((u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) - \theta^{(t)} \cdot (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)}))$

$s.t.\ v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0, \forall t$

$v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} \geq 1, \forall t$

$\Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + \eta_4^{(t)} \cdot u_o^{(t)}) = 0$ (9.23)

参数化模型（产出导向，$\phi^{(t)}$是模型参数）：

$D^* = \max \Sigma(\phi^{(t)} \cdot (u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) - (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)}))$

$s.t.\ v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0, \forall t$

$u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} \geq 1, \forall t$

$\Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + \eta_3^{(t)} \cdot u_o^{(t)}) = 0$ (9.24)

以投入导向为例，设置模型(9.23)的参数 $\theta^{(t)}$等于节点独立模型(9.22)得出的节点效率值 $E^{(t)*}$，求解模型，若模型的最优解目标函数值 $D^* = 0$，说明网络为**无损协同**；如果 $D^* < 0$ （D^*必定是 ≤ 0），说明网络为**有损协同**。

也可以求解参数化模型(9.23)的对偶模型：

$D^* = \max \Sigma(\alpha^{(t)})$

s.t. $X^{(t)}\cdot\lambda^{(t)} \le x_o^{(t)}\cdot(\theta^{(t)} - \alpha^{(t)})$, $\forall t$

$Z^{(in)(t)}\cdot\lambda^{(t)} \le z_o^{(in)(t)}\cdot(\theta^{(t)} - \alpha^{(t)} - \gamma)$, $\forall t$

$Y^{(t)}\cdot\lambda^{(t)} \ge y_o^{(t)}$, $\forall t$

$Z^{(out)(t)}\cdot\lambda^{(t)} \ge z_o^{(out)(t)}\cdot(1 - \gamma)$, $\forall t$

$\gamma \in R$, $\alpha^{(t)} \ge 0$ (9.25)

VRS 模型：

如果节点(t)有产出 $y^{(t)}$： $\mathbf{1}\cdot\lambda^{(t)} = 1$, $\forall t$

如果节点(t)没有产出 $y^{(t)}$： $\mathbf{1}\cdot\lambda^{(t)} = 1 - \gamma$, $\forall t$

9.3.3 系统效率的计算方法

以投入导向模型为例：

1) 模型**无损协同**($D^* = 0$)

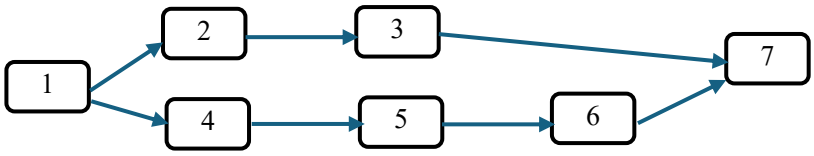
独立节点模型得出的节点效率值 $E^{(t)*}$与网络模型计算得出的节点效率值相等：

在模型可以实现**无损协同**时，网络模型的目标函数可以是节点效率的任意单调递增函数 $f\left(E^{(1)},\dots,E^{(N)}\right)$，节点独立最优解都可以组合构造出网络模型的最优解。据此，我们定义如下具有普适性的系统整体效率计算方法（幂乘积形式）：

$$\boldsymbol{E^* = \prod_{t=1}^{N}(E^{(t)*})^{\omega^{(t)}},\ 1 \le \sum \omega^{(t)} \le N}$$

$\omega^{(t)}$反映每个节点在网络中的角色：**如果 $\omega^{(t)} = 1, \forall t$，则系统效率为所有节点效率之乘积；如果 $\omega^{(t)} = 1/N$，则系统效率为所有节点效率之几何平均；如果各节点的 $\omega^{(t)}$不相同，但 $\Sigma\omega^{(t)} = 1$，则系统效率为所有节点效率之加权几何平均。**

乘积形式适用于串联结构，任意子阶段的低效率（$E^{(t)*} < 1$）都会以乘法形式拉低系统整体效率；几何平均形式适用于并联结构；对于包含复杂串联和并联的网络结构，以下是一个确定节点权重的简单示例：



(1) 先按照串联关系为节点分配初始权重 1：

$\omega^{(1)} = 1$， $\omega^{(2)} = 1$， $\omega^{(3)} = 1$， $\omega^{(4)} = 1$， $\omega^{(5)} = 1$， $\omega^{(6)} = 1$， $\omega^{(7)} = 1$。

(2) 分支一（节点 2 和 3）与分支二（节点 4、5、6）是并联关系，对两个分支取几何平均（其内部节点的权重在原来的基础上除以 2）：

$\omega^{(1)} = 1$，$\omega^{(2)} = 0.5$，$\omega^{(3)} = 0.5$，$\omega^{(4)} = 0.5$，$\omega^{(5)} = 0.5$，$\omega^{(6)} = 0.5$，$\omega^{(7)} = 1$。

$\sum \omega^{(t)} = 4.5$，处于几何平均（$\sum \omega^{(t)} = 1$）和乘积（$\sum \omega^{(t)} = 7$）之间。

以上只是根据网络结构特征得出的权重，实证研究中，还可以结合节点对系统的重要程度确定最终的权重。

2) 模型**有损协同**($D^* < 0$)

如果 $D^* < 0$，则说明网络模型为有损协同，有损协同的节点效率值为

$E^{(t)-} = (u^{(t)*} \cdot y_o^{(t)} + w^{(out)(t)*} \cdot z_o^{(out)(t)} - u_o^{(t)*}) / (v^{(t)*} \cdot x_o^{(t)} + w^{(in)(t)*} \cdot z_o^{(in)(t)}) \leq E^{(t)*}$，

如果求解的是对偶模型(9.25)，则 $E^{(t)-} = E^{(t)*} - \alpha^{(t)*} \leq E^{(t)*}$。

有损协同的系统效率值为

$$E^- = \prod_{t=1}^{N} (E^{(t)-})^{\omega^{(t)}},\ 1 \leq \sum \omega^{(t)} \leq N$$

有损协同得出的节点效率值**不具有唯一性**。

3) 有损协同的解决方法

一种解决方法是综合运用“权重比值约束模型（又称为锥比率 DEA，约束锥 DEA，保证域 DEA）”来阻止一项或多项中间流的权重为 0，恰当地设置权重比值约束，可以实现无损协同。

另一种方法是采用后面介绍的“基于全维有效面(FDEF)的广义网络 DEA”，该模型的所有权重均严格大于 0，只需满足一个简单的条件（充分非必要条件），就能够实现无损协同：网络存在只生产中间流不消耗中间流（只有 $z^{(out)}$没有 $z^{(in)}$）的节点，同时也存在不生产中间流只消耗中间流（只有 $z^{(in)}$没有 $z^{(out)}$）的节点。前面的两阶段网络模型是满足这个条件的。

9.3.4 与前面的两阶段模型的兼容性

前面介绍的两阶段网络模型是广义网络模型在两阶段结构时的特例，两阶段模型满足**无损协同**的充分条件，如果采用乘积方式计算系统整体效率，虽然二者的最优解（权重 v, u, w）不相同，但得出的节点效率和系统效率相同。

9.4 基于全维有效面(FDEF)的广义网络 DEA

经典 DEA 模型常被提到的一个缺陷是效率测量没有包含通常会在模型中出现的投入和产出松弛，基于这个原因出现了各类直接基于松弛测量效率的非径向模型。模型中投入和产出松弛问题并非经典 DEA 本身的缺陷，而是数据样本自身没有能够建立覆盖所有样本投影区域的强有效前沿，造成径向投影点落在与坐标轴平行的弱有效前沿上，如下图所示，DMU A 的投影点为 A_1，位于与纵坐标轴平行的弱有效前沿上，投入 x_2的松弛为 A_1B。各类非径向模型的投影点则位于强有效前沿 BC 段。非径向模型虽然解决了经典模型效率测量没有包含松弛的问题，但带来了新的问题：一是削弱了经典模型的对偶乘数模型的经济学意义，二是投影点往往并不符合逻辑，例如加性模型和 SBM 模型是以最不利于被评价 DMU 的“最远距离”来确定投影点。以下图所示情况为例，在加性模型和 SBM 模型中，投影点是距离 A 更远的 C，而不是更近的 B。

经典包络模型中出现的松弛对应于乘数模型中出现权重为 0。还有一种处理经典模型中松弛问题的方案是全维有效面(FDEF)。在示例图中，前沿的 BC 段是一个 FDEF，基于 BC 仿射包（Affine Hull, 在二维坐标中是直线，在多维空间中是超平面）的径向投影点为 A_2。

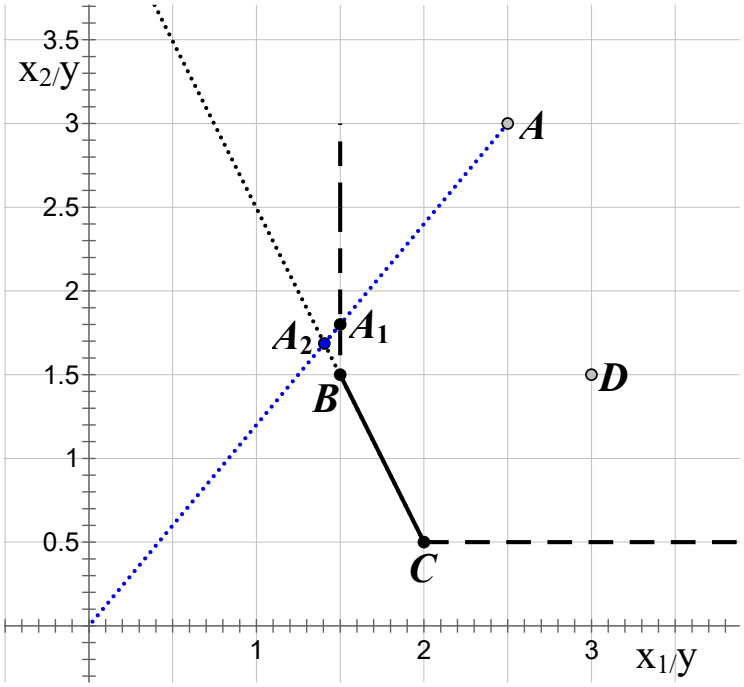


图 9-1 松弛处理方法示意图（两项投入一项产出 CRS 模型）

本节给出基于 FDEF 仿射包的广义网络 DEA 模型，可以消除经典模型中存在的松弛问题，包括投入和产出松弛，也包括中间流的松弛。中间流松弛的消失能够保证模型得出的第一阶段作为产出的中间流投影值完全等于第二阶段作为投入的中间流的投影值。

9.4.1 全维有效面(FDEF)简介

在数据包络分析(DEA)的凸包络几何理论中，全维有效面(Full-Dimensional Efficient Facets, FDEF)是刻画生产可能性集(PPS)前沿结构的核心概念之一。

1. 全维有效面(FDEF)的定义

在包含 m 个投入和 s 个产出的 DEA 模型中，生产可能性集 PPS $\subset \mathbb{R}^{m+s}$ 是一个多面体。如果该

多面体边界上的某一有效超平面与其相交构成的有效面(Efficient Facet)，其几何维度恰好等于 $m+s-1$（即空间维度的最高可能自由度），则该面被称为**全维有效面(FDEF)**(Olesen & Petersen, 1996, 2003)。从乘数（对偶）角度来看，支撑 FDEF 的超平面法向量所对应的所有投入乘数 v 与产出乘数 u 均严格大于零(即 $v>0,u>0$)，不存在任何零权重。

2. 基于 FDEF 仿射包的 DEA 模型的优点

基于 FDEF 约束或识别构建的 DEA 模型具有以下显著优势：

彻底消除零权重问题：传统 DEA 模型常因乘数退化出现“零权重”现象（即某些投入或产出被赋予零权重），而 FDEF 保证了所有指标在边际评价中均发挥作用。

提供合理的经济学解释（边际替代率）：由于 FDEF 的法向量全为正值，由其导出的边际替代率(MRS)均为有限且非零的确定值，避免了低维有效面(LDEF)上边际替代率为 0 或无穷大的经济学失效问题。

优化投影与目标设置：将非有效决策单元(DMU)投影至 FDEF 超平面上，可以确保改进目标具备强有效性，彻底消除松弛问题。

3. 求解 DEA 模型样本中所有 FDEF 的方法

目前求解和识别多面体前沿上所有 FDEFs 的方法主要分为两类：

基于凸包(Convex Hull)的计算几何算法：利用计算几何中的顶点枚举与极面生成算法，对 DMU 顶点集构建凸包并提取维数为 $m+s-1$ 且法向量大于 0 的超平面(Olesen & Petersen, 2003)。

基于混合整数规划(MIP)的迭代搜索法：利用对偶理论，通过 MIP 搜寻所有权重严格大于零的超平面 (Olesen & Petersen, 2003; Jahanshahloo, Lotfi, Rezai, & Balf, 2007; Jahanshahloo, Shirzadi, & Mirdehghan, 2009; Olyaie, Roshdi, Jahanshahloo, & Asgharian, 2014)。

4. 样本不存在 FDEF 的现象与前沿严重退化

如果某一数据集在求解时不存在任何 FDEF，几何上意味着该样本的包络前沿完全由低维有效面(Lower-Dimensional Efficient Facets, LDEF)构成。这在理论与应用中具有明确的警示意义，表明前沿严重退化(Severe Frontier Degeneracy)，数据存在严重的病态数据(Ill-conditioned data)或维度灾难（即样本量相对于投入产出指标维度严重不足），导致生产前沿在高维空间中“折叠”或“塌陷”到低维子空间(Olesen & Petersen, 1996)。

一个有效的解决方法是引入足够数量的标杆 DMU 作为模型的参考集。

样本中存在一个 FDEF 并不意味着能够建立满意的评价模型，如果模型只有一个或极少量的 FDEF，被评价 DMU 有可能投影到严重远离 FDEF 的区域（位于 FDEF 的支撑超平面上，但严重远离 FDEF），得出不切实际的目标值。如何判断样本中的 FDEF 是否充足，是一个需要在理论和实践中继续探讨的问题。判断投影值（目标值）是否超出合理范围是判断 FDEF 是否充足的一个实用方法。

9.4.2 基于 FDEF 仿射包的广义网络 DEA 模型

基于 FDEF 仿射包的 DEA 模型就是把模型的权重限定为来自 FDEF 超平面集合，基于 FDEF 仿射包的广义网络 DEA 模型的表达式为：

$$E^* = \max \Sigma((u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) / (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)})) / N$$

$$s.t.\ \Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + (\eta_3^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) = 0$$

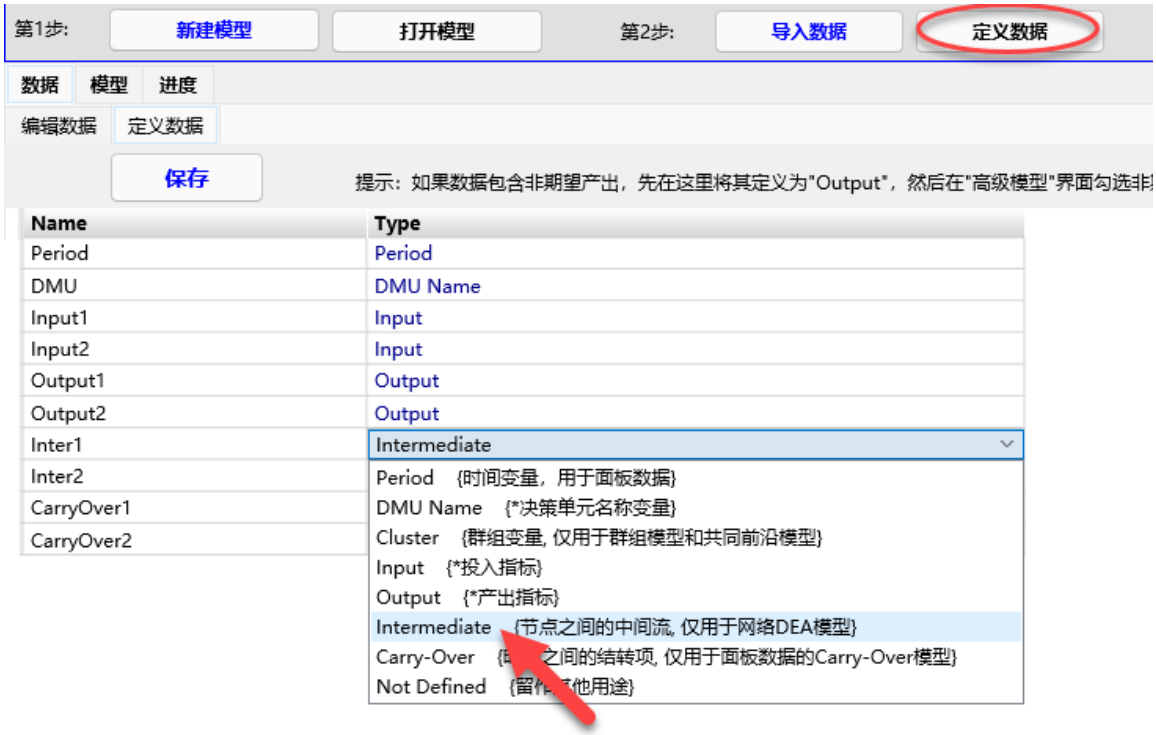
$$(v^{(t)}, u^{(t)}, w^{(in)(t)}, w^{(out)(t)}, u_o^{(t)}) \in F^{(t)} \qquad (9.26)$$

$F^{(t)}$表示阶段 t 的 FDEF 超平面集合。

基于 FDEF 仿射包的广义网络 DEA 模型，由于其权重全部严格大于 0，只需满足一个简单的条件（充分非必要条件），就能够实现无损协同：网络存在只生产中间流不消耗中间流（只有 $z^{(out)}$没有 $z^{(in)}$）的节点，同时也存在不生产中间流只消耗中间流（只有 $z^{(in)}$没有 $z^{(out)}$）的节点。

9.5 网络 DEA 的软件操作演示

9.5.1 定义数据



9.5.2 设置 DEA 模型的常规选项

生产函数类型（经典前沿生产函数或 FDEF 仿射包前沿生产函数）、模型导向和规模报酬。

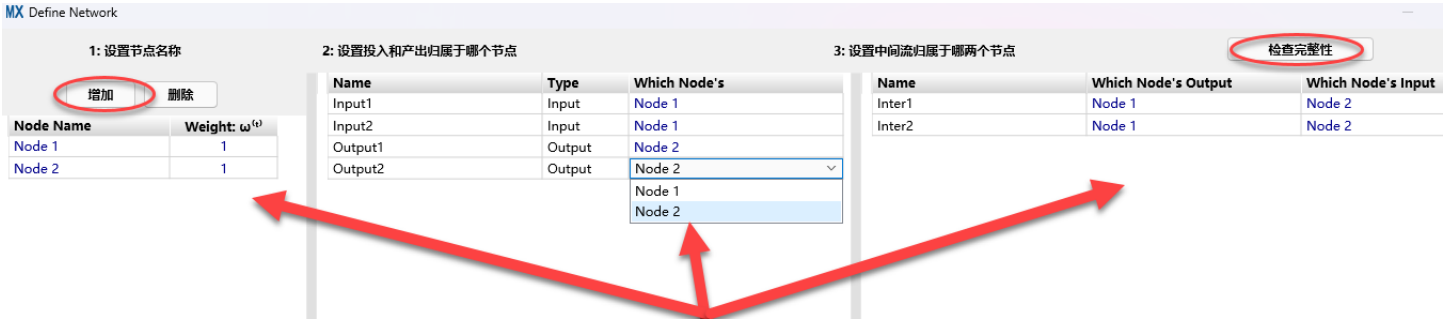
9.5.3 设置其他模型选项

选择“**两阶段网络 DEA**”或“**广义网络 DEA**”。

网络 DEA 可以与所有“**面板数据模型**”进行组合，包括：Malmquist TFP 指数、HMB TFP 指数、窗口 DEA 和 Carry-Over DEA。

9.5.4 定义网络结构

如果选择“广义网络 DEA”，还需要定义网络结构。广义网络模型各节点的初始默认权重 $\omega^{(t)}=1$，即系统效率为节点效率之乘积；如果需要系统效率为节点效率之几何平均，请设置各节点的权重之和等于 1，即 $\Sigma\omega^{(t)}=1$。



9.5.5 输出结果

输出结果包括系统整体效率值、投入产出和中间流的权重、投入产出和中间流的投影值（目标值）、节点的效率值、节点的参考标杆及其系数(Lambda)等，VRS 模型还包括规模报酬常数项($u_o^{(t)}$)，广义网络模型还包括 D*（用于判断网络模型给出的节点效率值是否具有唯一性）。

附录：CRS 两阶段网络 DEA 效率乘积分解证明

两阶段网络 DEA 模型(9.6)，被评价 DMU 的整体网络效率 **E*** 等于阶段 1 模型(9.1)和阶段 2 模型(9.2)求解出的最优阶段效率 $\mathbf{E^{(1)*}}$ 与 $\mathbf{E^{(2)*}}$ 之积：

$$\mathbf{E^* = E^{(1)*} \times E^{(2)*}}$$

一、**证明 $e^* \le e^{(1)*} \times e^{(2)*}$**

设 $(v', w^{(1)\prime}, w^{(2)\prime}, u')$ 为系统整体模型的**任意一组可行解**。

利用网络模型的中间约束 $w^{(1)\prime} z_0 = w^{(2)\prime} z_0$，可将该可行解对应的目标函数值分解为：

$$\frac{u'y_0}{v'x_0} = \frac{w^{(1)\prime} z_0}{v'x_0} \cdot \frac{u'y_0}{w^{(2)\prime} z_0}$$

根据可行解的定义：

1. 向量对 $(v', w^{(1)\prime})$ 满足 $\frac{w^{(1)\prime} z_j}{v'x_j} \le 1\ (\forall j)$，因而 $(v', w^{(1)\prime})$ 是阶段 1 模型的一组可行解。根据 $e^{(1)*}$ 的最大值定义：

$$\frac{w^{(1)\prime} z_0}{v'x_0} \le e^{(1)*}$$

2. 向量对 $(w^{(2)\prime}, u')$ 满足 $\frac{u'y_j}{w^{(2)\prime} z_j} \le 1\ (\forall j)$，因而 $(w^{(2)\prime}, u')$ 是阶段 2 模型的一组可行解。根据 $e^{(2)*}$ 的最大值定义：

$$\frac{u'y_0}{w^{(2)\prime} z_0} \le e^{(2)*}$$

将上述两个不等式代入乘积分解式可得：

$$\frac{u'y_0}{v'x_0} \le e^{(1)*} \times e^{(2)*}$$

由于上式对网络模型可行域内的**任意**可行解均成立，因此对可行域上的最大值(即最优解)亦成立：

$$e^* \le e^{(1)*} \times e^{(2)*}$$

二、**证明 $e^* \ge e^{(1)*} \times e^{(2)*}$**

设 $(v^*, w^{(1)*})$ 为阶段 1 模型的一组**最优解**，满足：

$$e^{(1)*} = \frac{w^{(1)*} z_0}{v^* x_0}$$

设 $(w^{(2)*}, u^*)$ 为阶段 2 模型的一组**最优解**，满足：

$$e^{(2)*} = \frac{u^* y_0}{w^{(2)*} z_0}$$

构造一个正标量缩放因子：

$$\alpha = \frac{w^{(1)*} z_0}{w^{(2)*} z_0} > 0$$

利用阶段 2 模型分式规划的齐次性(即同比例放大分子分母权重不改变约束与比值)，对阶段 2 最优解进行线性缩放，构造新的权重向量：

$$\tilde{w}^{(2)} = \alpha w^{(2)*}, \quad \tilde{u} = \alpha u^*$$

验证组合解 $(v^*, w^{(1)*}, \tilde{w}^{(2)}, \tilde{u})$ 在网络模型中的可行性：

1. **阶段 1 约束**：$\frac{w^{(1)*} z_j}{v^* x_j} \le 1\ (\forall j)$ 显然满足。
2. **阶段 2 约束**：对于任意 j，有：

$$\frac{\tilde{u} y_j}{\tilde{w}^{(2)} z_j} = \frac{\alpha u^* y_j}{\alpha w^{(2)*} z_j} = \frac{u^* y_j}{w^{(2)*} z_j} \le 1$$

满足约束条件。

3. **中间流等式约束**：

$$\tilde{w}^{(2)} z_0 = (\alpha w^{(2)*}) z_0 = \left(\frac{w^{(1)*} z_0}{w^{(2)*} z_0}\right)(w^{(2)*} z_0) = w^{(1)*} z_0$$

完美满足 $w^{(1)} z_0 = w^{(2)} z_0$ 的耦合约束。

因此，$\left(v^*, w^{(1)*}, \tilde{w}^{(2)}, \tilde{u}\right)$ 是网络模型的一组**有效可行解**。

计算该可行解在网络模型中的目标函数值：

$$\frac{\tilde{u}y_0}{v^*x_0} = \frac{\alpha u^* y_0}{v^* x_0} = \frac{u^* y_0}{w^{(2)*} z_0} \cdot \frac{w^{(1)*} z_0}{v^* x_0} = e^{(2)*} \times e^{(1)*}$$

由于 e^* 是网络模型的全局最大值，而上述组合解仅为该模型可行域中的一个特定解，因此必有：

$$e^* \geq e^{(1)*} \times e^{(2)*}$$

三、结论

综合上界与下界的不等式：

$$e^* \leq e^{(1)*} \times e^{(2)*} \quad \text{且} \quad e^* \geq e^{(1)*} \times e^{(2)*}$$

根据实数大小关系的反对称性，必然推出：

$$e^* = e^{(1)*} \times e^{(2)*}$$

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Chapter 9 Innovative Two-Stage and Generalized Network DEA

As the core non-parametric tool for evaluating the relative efficiency of decision-making units (DMUs), traditional data envelopment analysis (DEA) treats each DMU as a “black box” with an invisible internal structure, focusing only on the transformation from final inputs to final outputs. Real production systems, however, often contain complex sub-processes, multi-stage structures, and intermediates flowing among them. Network DEA was proposed precisely to open this “black box” and to analyze the operating efficiency of each node (stage) within the system and its impact on the system as a whole. In the technical logic of network DEA, the individual production stages are explicitly independent of one another technologically; that is, each node constructs its own production possibility set (PPS). This technical independence makes it feasible and legitimate to form linear combinations of the Stage 1 technology of one DMU with the Stage 2 technology of another.[i]

Although two-stage and generalized network DEA (covering multiple nodes, with serial and parallel structures, and arbitrary topologies) greatly extends the applicability of conventional DEA, it has long faced a pain point at the theoretical and algorithmic level: **the non-uniqueness of the system efficiency decomposition, and the fact that the node efficiency values obtained from the decomposition are often not the efficiency values the nodes would attain under their own stand-alone PPS**. In solving the multiplier model of conventional network DEA, multiple optimal solutions may exist, so the algebraic mapping between the overall system efficiency and the efficiencies of the sub-nodes is unstable. Such an unstable decomposition obstructs further analysis of the node efficiency values of network DEA models (e.g., TFP index analysis at the node level). To address this problem, previous studies have mostly relied on artificially imposing secondary objectives, introducing subjective preference weights, or constructing leader–follower game models to forcibly “lock in” the decomposition results.

The innovative two-stage network DEA model developed by the MaxDEA team fundamentally resolves this difficulty: the system efficiency and the two stage efficiencies naturally satisfy a unique equality relationship: **The overall system efficiency equals the product of the two unique stage efficiencies.** On this basis, it is extended to “generalized network DEA” and “generalized network DEA based on full-dimensional efficient facets (FDEF)”.

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[i] In this chapter, "Stage" is used in the two-stage network DEA, and "Node" is used in the generalized network DEA, to represent the internal structure of the network. In addition, for the weight vectors v and u, "weight" and "(shadow) price" are used interchangeably.

Notation and Data Description

Formulas are written in matrix and vector notation. For the convenience of readers who wish to copy the formulas, **plain-text format** is used in most cases.

Matrices: X, Y, and Z are the input, output, and intermediate matrices. In multiplier models the number of rows of the matrices equals the number of sample DMUs; in envelopment models the number of columns equals the number of sample DMUs.

X, Y, Z > 0.

Column vectors: x_o, y_o, z_o denote the input, output, and intermediate vectors of the DMU under evaluation (the subscript $_o$ is used for better visual prominence). λ is the vector of linear-combination coefficients of the DMUs in the envelopment model.

Row vectors: v, u, and w are the weights of the inputs, outputs, and intermediates in multiplier models.

v, u, w, and λ are non-negative variables (≥ 0) by default.

Bold **1** denotes a row or column vector whose elements are all 1.

$u_o \in R$ is a scalar variable representing the returns-to-scale constant term (the intercept) in the VRS multiplier model.

Superscripts: $^{(1)}$ $^{(2)}$ $^{(t)}$ denote Stage 1, Stage 2, and node t; $^{(in)}$ $^{(out)}$ indicate whether the intermediates act as inputs or outputs at node t.

9.1 Two-Stage Network DEA (Constant Returns to Scale, CRS)

9.1.1 The Two-Stage Network DEA Model

Kao & Hwang (2008) and Liang, Cook, & Zhu (2008) proposed two-stage network DEA models that define the overall system efficiency as the product of the efficiencies of the two nodes, initiating the multiplicative efficiency decomposition framework (Multiplicative Decomposition) for two-stage network DEA models.

Efficiency of Stage 1

$$E^{(1)}* = \max (w \cdot z_o / v \cdot x_o)$$

$$\text{s.t.} \quad (w \cdot Z) / (v \cdot X) \le 1 \qquad (9.1)$$

Efficiency of Stage 2

$$E^{(2)}* = \max (u \cdot y_o / w \cdot z_o)$$

$$\text{s.t.} \ (u \cdot Y) / (w \cdot Z) \le 1 \qquad (9.2)$$

Overall system efficiency

$$E* = E^{(1)} \times E^{(2)} = \max (u \cdot y_o / w \cdot z_o) \times (w \cdot z_o / v \cdot x_o) = \max (u \cdot y_o / v \cdot x_o)$$

$$\text{s.t.} \quad (w \cdot Z) / (v \cdot X) \le 1$$

$$(u \cdot Y) / (w \cdot Z) \le 1 \qquad (9.3)$$

Applying the Charnes-Cooper transformation yields the following linear programming (LP) model (input-oriented):

$$\max u \cdot y_o$$

$$\text{s.t.} \ v \cdot X - w \cdot Z \ge 0$$

$$w \cdot Z - u \cdot Y \ge 0$$

$$v \cdot x_o = 1 \qquad (9.4)$$

Dual envelopment model

$$\min \theta$$

$$\text{s.t.} \ X \cdot \lambda^{(1)} \le x_o \cdot \theta$$

$$Y \cdot \lambda^{(2)} \ge y_o$$

$$Z \cdot \lambda^{(1)} \ge Z \cdot \lambda^{(2)}$$

$$\theta \in R \qquad (9.5)$$

The overall system efficiency of two-stage network DEA can be decomposed into the product of two

components:

$E^* = (u \cdot y_o) / (v \cdot x_o) = (u \cdot y_o) / (w \cdot z_o) \cdot (w \cdot z_o) / (v \cdot x_o) = E^{(1)} \times E^{(2)} \le E^{(1)*} \times E^{(2)*}$

By opening the "black box" of the system, the above two-stage network DEA framework successfully captures the flow of intermediate products within the system. Within this theoretical architecture, however, the model exhibits the following features and defects:

Feature 1: rigid synergy with equal intermediate prices (Rigid Synergy)

The multiplier model strictly requires that the weights $w^{(out)}$ of the Stage 1 intermediates (as outputs) and the weights $w^{(in)}$ of the Stage 2 intermediates (as inputs) be strictly equal on a one-to-one basis.

Feature 2: hybrid nature of the technology improvement path

When mapped into the dual envelopment model, the rigid synergy mechanism appears as a mixture of radial and non-radial adjustment: the model imposes radial proportional scaling constraints (θ) only on the initial input x (or the final output y in output-oriented models), while the intermediates z are granted complete freedom. This hybrid structure weakens the economic meaning of the model and causes the projections to deviate from the classical technology improvement path.

Defect 1: non-uniqueness of the system efficiency decomposition

Under this framework, although the overall system efficiency is uniquely determined, the stage efficiency values obtained from the decomposition may be not unique, because the multiplier model often has multiple optimal solutions.

$E^* = E^{(1)} \times E^{(2)}$, $E^{(1)}$ and $E^{(2)}$ are not unique.

Defect 2: the decomposed stage efficiency values may deviate from their respective PPS

The stage efficiencies obtained under the rigid synergy mechanism may not the true technical efficiencies of the stages under their own PPS.

$E^{(1)} \le E^{(1)*}$, $E^{(2)} \le E^{(2)*}$.

9.1.2 Two-Stage Network DEA Model Based on Conservation of the Virtual Value of Intermediates

To address the pain points of the above framework, the following two-stage network DEA model with flexible synergy, which requires only that the total value of the intermediates be equal, is proposed:

$E^* = \max (u \cdot y_o) / (v \cdot x_o)$

s.t. $(w^{(1)} \cdot Z) / (v \cdot X) \le 1$

$(u \cdot Y) / (w^{(2)} \cdot Z) \le 1$

$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o}$ (9.6)

Applying the Charnes-Cooper transformation yields the following linear programming (LP) model (input-oriented):

$E^* = \max u \cdot y_o$
s.t. $v \cdot X - w^{(1)} \cdot Z \geq 0$
$w^{(2)} \cdot Z - u \cdot Y \geq 0$
$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o}$
$v \cdot x_o = 1$ (9.7)

Dual envelopment model

$E^* = \min \theta$

s.t. $X \cdot \lambda^{(1)} \leq x_o \cdot \theta$

$\mathbf{Z \cdot \lambda^{(1)} \geq z_o \cdot \tau}$

$\mathbf{Z \cdot \lambda^{(2)} \leq z_o \cdot \tau}$

$Y \cdot \lambda^{(2)} \geq y_o$

$\theta, \tau \in R$ (9.8)

Alternatively, applying the Charnes-Cooper transformation yields the following linear programming (LP) model (output-oriented):

$1/E^* = \min v \cdot x_o$
s.t. $v \cdot X - w^{(1)} \cdot Z \geq 0$
$w^{(2)} \cdot Z - u \cdot Y \geq 0$
$\mathbf{w^{(1)} \cdot z_o = w^{(2)} \cdot z_o}$
$u \cdot y_o = 1$ (9.9)

Dual envelopment model

$1/E^* = \max \phi$

s.t. $X \cdot \lambda^{(1)} \leq x_o$

$\mathbf{Z \cdot \lambda^{(1)} \geq z_o \cdot \tau}$

$\mathbf{Z \cdot \lambda^{(2)} \leq z_o \cdot \tau}$

$Y \cdot \lambda^{(2)} \geq y_o \cdot \phi$

$\phi, \tau \in R$ (9.10)

The two-stage network DEA model based on conservation of the virtual value of intermediates has the following properties:

1) Flexible synergy of intermediates

It abandons the strong exogenous constraint that the price of each intermediate must be equal, and instead requires only that all intermediates of the DMU under evaluation satisfy "equal total value" as they flow within the system. This matches the circulation characteristic that internal settlement of intermediates generates no externalities, and allows the intermediate weights (shadow prices) to be determined within the system according to the classical DEA optimization principle.

$$w^{(1)} \cdot z_o = w^{(2)} \cdot z_o$$

2) Purely classical technology improvement path

It fully adopts the classical radial proportional technology improvement path, completely preserving the economic meaning of the classical DEA model.

$$Z \cdot \lambda^{(1)} \geq z_o \cdot \tau$$

$$Z \cdot \lambda^{(2)} \leq z_o \cdot \tau$$

3) The system efficiency decomposition is unique, and the decomposed stage efficiency values are the technical efficiencies under each stage's own PPS.

9.1.3 Efficiency Decomposition of the Two-Stage Network DEA Model (CRS)

In the two-stage network DEA model, the overall network efficiency obtained from model (9.6) equals the product of the efficiency of stage 1 obtained from model (9.1) and the efficiency of stage 2 obtained from (9.2) **(the proof is given in the appendix)**:

$$E^* = E^{(1)*} \times E^{(2)*}$$

9.1.4 The Flexible Synergy Mechanism for Intermediates

In network DEA models, the shadow prices determined independently by each stage are valid only within that stage, and they can be scaled up or down as a whole without loss of equivalence. Since the production technologies of the two stages are independent of each other, a synergy mechanism is needed when evaluating the overall system efficiency, in order to reflect the bridging role of the intermediates in the network structure and the rules governing their flows.

Flexible synergy abandons the stringent constraint of rigid synergy (one-to-one equality of intermediate prices, $w^{(1)} = w^{(2)}$) , and relaxes it to "equality of the total value of the intermediates between the production side and the consumption side within the system" ($w^{(1)} \cdot z_o = w^{(2)} \cdot z_o$). It is precisely this core change that achieves a unique decomposition of system efficiency and keeps the decomposed stage efficiencies consistent with the stand-alone technical efficiencies of the stages.

Analysis from the Perspective of Shadow Prices and Production Function Theory

In duality and linear programming theory, the multipliers can be directly interpreted as the internal shadow prices or marginal valuations of the inputs, outputs, and intermediates.

Rigid synergy forces $w^{(in)} = w^{(out)}$, imposing a unified cross-stage "exogenous market price" on the intermediate products. However, Stage 1 and Stage 2 embody different production technologies, and the marginal rates of substitution (MRS) of the intermediates differ between the two stages. Forcing the shadow prices of the intermediates to be equal one-to-one across the two stages means fitting two independent convex frontiers with a single linear tangent plane, which inevitably causes the fitted frontier to deviate from the frontiers of the two stages, so that the stage efficiencies are distorted by the rigid synergy constraint.

Flexible synergy only requires value conservation when the intermediates are transferred within the system, which grants $w^{(1)}$ and $w^{(2)}$ independent degrees of freedom so that each can precisely match the normal vector of the optimal frontier of its own stage (i.e., each stage's independent local optimal shadow prices). As a result, the stage technical efficiencies evaluated by the network system are consistent with the technical efficiencies evaluated independently for each stage.

Analysis from the Perspective of Weighted Virtual Input-Output Theory

The core idea of classical DEA is to aggregate multiple inputs and multiple outputs into scalar virtual inputs and virtual outputs, and—under the constraint that the ratio of virtual output to virtual input does not exceed 1—to give the DMU under evaluation complete freedom to seek optimal weights.

Intermediates play a dual role: they are the virtual output of Stage 1 and the virtual input of Stage 2. Rigid synergy requires a single weight vector to match both input-output structures simultaneously, which restricts each stage's freedom in optimization and prevents the model from reaching the stages' optimal solutions. Flexible synergy removes this over-constraint and imposes only the constraint that the virtual aggregated value of the intermediates at their producing side be equal to that at their consuming side. This ensures that the virtual value passed from Stage 1 to Stage 2 is neither created out of nothing nor destroyed without reason—perfectly consistent with the closed-loop characteristic that intermediate products exist only inside the network system and generate no externalities—while retaining the complete freedom of the DMU under evaluation to seek the optimal weights of each stage, so that each stage can still attain its own optimal solution under the network DEA architecture.

Table 9-1 Comparison of the constraints on intermediates between rigid synergy and flexible synergy

	Rigid Synergy	**Flexible Synergy**
Multiplier model	**Rigid** The prices of the intermediates are equal one-to-one across the two stages $w^{(1)} = w^{(2)}$	**Flexible** The total value of the intermediates is equal across the two stages $w^{(1)} \cdot z_o = w^{(2)} \cdot z_o$

Envelopment model	**Free scaling** Only the circulation logic must be observed $z_i \cdot \lambda^{(2)} \le \hat{z}_i \le z_i \cdot \lambda^{(1)}, \forall_i$ Each intermediate is scaled freely and independently	**Radial scaling** Observing the circulation logic and the intermediate radial changes $z_i \cdot \lambda^{(2)} \le z_{io} \cdot \tau \le z_i \cdot \lambda^{(1)}, \forall_i$ Or, expressed as $z_i \cdot \lambda^{(2)} \le \hat{z}_i \le z_i \cdot \lambda^{(1)}, \forall_i$ $\hat{z}_i = z_{io} \cdot \tau$ τ is the scaling scalar of the intermediates

* When there is only one intermediate in the network, the two synergy mechanisms are equivalent: $w^{(1)} \cdot z_o = w^{(2)} \cdot z_o \rightarrow w^{(1)} = w^{(2)}$.

9.1.5 Why the Efficiency Decomposition of Two-Stage Network Models Takes the Product Form

The multiplicative structure of the two-stage system efficiency decomposition avoids excessive substitution compensation, in which a highly efficient stage would overly compensate for an inefficient one. “Neither stage may fall behind” is an inherent requirement of a two-stage network system. The product form of efficiency embodies the typical property of a series system: the inefficiency of any sub-stage ($E^{(t)*} < 1$) drags down the overall system efficiency multiplicatively.

For example, two companies A and B produce the same product from the same raw materials. In Stage 1, company A uses 100 units of raw materials to produce 100 units of intermediate products, while company B uses 100 units of raw materials to produce 10 units; compared with A, B’s Stage 1 efficiency is 0.1. In Stage 2, company A uses 100 units of intermediates to produce 100 units of final products, while company B uses 10 units of intermediates to produce 9 units; compared with A, B’s Stage 2 efficiency is 0.9. The product form correctly reflects this series-chain effect: B’s overall efficiency is 0.09, not the 0.5 given by the arithmetic average form.

9.2 Two-Stage Network DEA (Variable Returns to Scale, VRS)

9.2.1 Model Definitions

1) Input-oriented model (the intercept is assigned to the output side):

Stage 1 model:

$E^{(1)}* = \max (w^{(1)} \cdot z_o - u_o^{(1)}) / (v \cdot x_o)$
$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$ (9.11)

Stage 2 model:

$E^{(2)}* = \max (u \cdot y_o - u_o^{(2)}) / (w^{(2)} \cdot z_o)$

$s.t.\ w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$ (9.12)

Two-stage network model:

$E* = \max (w^{(1)} \cdot z_o - u_o^{(1)}) / (v \cdot x_o) \cdot (u \cdot y_o - u_o^{(2)}) / (w^{(2)} \cdot z_o) = \max (u \cdot y_o - u_o^{(2)}) / (v \cdot x_o)$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$
$w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$
$(w^{(1)} \cdot z_o - u_o^{(1)}) = w^{(2)} \cdot z_o$ (9.13)

Applying the Charnes-Cooper transformation converts it into a linear program.

$E* = \max (u \cdot y_o - u_o^{(2)})$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$
$w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$
$v \cdot x_o = 1$
$w^{(1)} \cdot z_o - u_o^{(1)} = w^{(2)} \cdot z_o$ (9.14)

Dual envelopment model:

$E* = \min \theta$

$s.t.\ X \cdot \lambda^{(1)} \leq x_o \cdot \theta$

$Z \cdot \lambda^{(1)} \geq z_o \cdot \tau$

$Z \cdot \lambda^{(2)} \leq z_o \cdot \tau$

$Y \cdot \lambda^{(2)} \geq y_o$

$\mathbf{1} \cdot \lambda^{(1)} = \tau$

$\mathbf{1} \cdot \lambda^{(2)} = 1$

$\theta, \tau \in R$ (9.15)

2) Output-oriented model (the intercept is assigned to the input side):

Stage 1 model:

$E^{(1)}* = \max (w^{(1)} \cdot z_o - u_o^{(1)}) / (v \cdot x_o)$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$ (9.16)

Stage 2 model:

$E^{(2)}* = \max (u \cdot y_o - u_o^{(2)}) / (w^{(2)} \cdot z_o)$

$s.t.\ w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$ (9.17)

Two-stage network model:

$E* = \max (w^{(1)} \cdot z_o) / (v \cdot x_o + u_o^{(1)}) \cdot (u \cdot y_o) / (w^{(2)} \cdot z_o + u_o^{(2)}) = \max (u \cdot y_o) / (v \cdot x_o + u_o^{(1)})$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$

$w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$

$w^{(1)} \cdot z_o = w^{(2)} \cdot z_o + u_o^{(2)}$ (9.18)

Applying the Charnes-Cooper transformation converts it into a linear program.

$1/E* = \min (v \cdot x_o + u_o^{(1)})$

$s.t.\ v \cdot X - w^{(1)} \cdot Z + u_o^{(1)} \geq 0$

$w^{(2)} \cdot Z - u \cdot Y + u_o^{(2)} \geq 0$

$u \cdot y_o = 1$

$w^{(1)} \cdot z_o = w^{(2)} \cdot z_o + u_o^{(2)}$ (9.19)

Dual envelopment model:

$1/E* = \max \phi$

$s.t.\ X \cdot \lambda^{(1)} \leq x_o$

$Z \cdot \lambda^{(1)} \geq z_o \cdot \tau$

$Z \cdot \lambda^{(2)} \leq z_o \cdot \tau$

$Y \cdot \lambda^{(2)} \geq \phi \cdot y_o$

$\mathbf{1} \cdot \lambda^{(1)} = 1$

$\mathbf{1} \cdot \lambda^{(2)} = \tau$

$\phi, \tau \in R$ (9.20)

9.2.2 Efficiency Decomposition

In the two-stage network DEA model, the overall network efficiency obtained from model (9.13) equals the product of the efficiency of stage 1 obtained from the model (9.11) and the efficiency of stage 2 obtained from model (9.12) (The proof is similar to that of the CRS model):

$$\mathbf{E* = E^{(1)}* \times E^{(2)}*}$$

9.3 Generalized Network DEA Model

9.3.1 Model Definition

$E^* = \max \Sigma((u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) / (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)})) / N$

s.t. $v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0, \forall t$

$\Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + (\eta_3^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) = 0$ (9.21)

N is the number of nodes in the network

η indicates how the intercept term is assigned in the VRS model: $\eta_1^{(t)} + \eta_2^{(t)} + \eta_3^{(t)} + \eta_4^{(t)} = 1, \eta \in \{0, 1\}, \forall t$

If node (t) has no input $x^{(t)}$: $\eta_1^{(t)} = 0$

If node (t) has no output $y^{(t)}$: $\eta_2^{(t)} = 0$

If node (t) has no intermediate flow $z^{(in)(t)}$: $\eta_3^{(t)} = 0$

If node (t) has no intermediate flow $z^{(out)(t)}$: $\eta_4^{(t)} = 0$

In the input-oriented model, the intercept is assigned to the output side: $\eta_2^{(t)} + \eta_4^{(t)} = 1$, if node (t) has output $y^{(t)}$, $\eta_2^{(t)} = 1$

In the output-oriented model, the intercept is assigned to the input side: $\eta_1^{(t)} + \eta_3^{(t)} = 1$, if node (t) has input $x^{(t)}$, $\eta_1^{(t)} = 1$

CRS model: all η are 0

9.3.2 Solving the Model

First solve the stand-alone node models to obtain each node's optimal efficiency value $E^{(t)*}$:

$E^{(t)*} = \max (u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) / (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)})$

s.t. $v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0$ (9.22)

The objective function of the generalized network model is a nonlinear program involving a sum of fractions, and it must allow the node weights to be scaled proportionally; it cannot be linearized via the Charnes-Cooper transformation. It can be solved by the Dinkelbach parametric approach.

Parametric model (input-oriented; $\theta^{(t)}$ are the model parameters):

$D^* = \max \Sigma((u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) - \theta^{(t)} \cdot (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)}))$

s.t. $v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0, \forall t$

$v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} \geq 1, \forall t$

$\Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + \eta_4^{(t)} \cdot u_o^{(t)}) = 0$ (9.23)

Parametric model (output-oriented; $\phi^{(t)}$ are the model parameters):

$D^* = \max \Sigma(\phi^{(t)} \cdot (u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)}) \cdot u_o^{(t)}) - (v^{(t)} \cdot x_o^{(t)} + w^{(in)(t)} \cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)}) \cdot u_o^{(t)}))$

s.t. $v^{(t)} \cdot X^{(t)} + w^{(in)(t)} \cdot Z^{(in)(t)} - u^{(t)} \cdot Y^{(t)} - w^{(out)(t)} \cdot Z^{(out)(t)} + u_o^{(t)} \geq 0, \forall t$

$u^{(t)} \cdot y_o^{(t)} + w^{(out)(t)} \cdot z_o^{(out)(t)} \geq 1, \forall t$

$\Sigma(w^{(in)(t)} \cdot z_o^{(in)(t)} - w^{(out)(t)} \cdot z_o^{(out)(t)} + \eta_3^{(t)} \cdot u_o^{(t)}) = 0$ (9.24)

Taking the input-oriented model as an example, set the model parameters $\theta^{(t)}$ for each node equal to its efficiency $E^{(t)*}$ obtained from the stand-alone node model (9.22). Solve the model; if the optimal objective function value $D^* = 0$, the network exhibits **lossless synergy**; if $D^* < 0$ (D^* must be ≤ 0), the network exhibits **lossy synergy**.

Alternatively, the dual model of the parametric model can be solved:

$D^* = \max \Sigma(\alpha^{(t)})$

s.t. $X^{(t)} \cdot \lambda^{(t)} \le x_o^{(t)} \cdot (\theta^{(t)} - \alpha^{(t)}), \forall t$

$Z^{(in)(t)} \cdot \lambda^{(t)} \le z_o^{(in)(t)} \cdot (\theta^{(t)} - \alpha^{(t)} - \gamma), \forall t$

$Y^{(t)} \cdot \lambda^{(t)} \ge y_o^{(t)}, \forall t$

$Z^{(out)(t)} \cdot \lambda^{(t)} \ge z_o^{(out)(t)} \cdot (1 - \gamma), \forall t$

$\gamma \in R, \alpha^{(t)} \ge 0$ (9.25)

VRS model:

If node (t) has output $y^{(t)}$: $\mathbf{1} \cdot \lambda^{(t)} = 1, \forall t$

If node (t) has no output $y^{(t)}$: $\mathbf{1} \cdot \lambda^{(t)} = 1 - \gamma, \forall t$

9.3.3 Method for Computing System Efficiency

Taking the input-oriented model as an example:

1) The model with **lossless synergy (**$D^* = 0$)

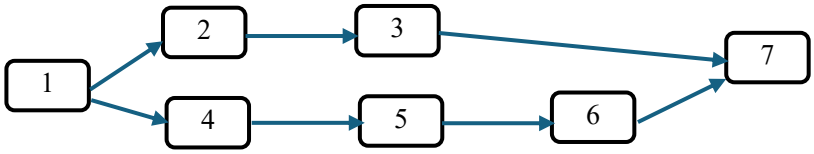
The node efficiency values obtained from the stand-alone node models $E^{(t)*}$ are equal to the node efficiency values computed by the network model:

When the model can achieve **lossless synergy**, the network model's objective function can be any monotonically increasing function of the node efficiencies $f(E^{(1)}, \dots, E^{(N)})$, and the stand-alone optimal solutions of the nodes can all be combined to construct an optimal solution of the network model. Accordingly, we define the following universal method for computing the overall system efficiency (power-product form):

$$\boldsymbol{E^* = \prod_{t=1}^{N} (E^{(t)*})^{\omega^{(t)}},\ 1 \le \sum \omega^{(t)} \le N}$$

$\omega^{(t)}$ reflects the role of each node in the network: **if $\omega^{(t)} = 1, \forall t$, then the system efficiency is the product of all node efficiencies; if $\omega^{(t)} = 1/N$, then the system efficiency is the geometric mean of all node efficiencies; if the $\omega^{(t)}$ values of the nodes are not identical, but $\Sigma\omega^{(t)} = 1$, then the system efficiency is the weighted geometric mean of all node efficiencies.**

The product form can be applied to purely series structures, where the inefficiency of any sub-stage ($E^{(t)*} < 1$) drags down the overall system efficiency multiplicatively; the geometric mean form can be applied to purely parallel structures. For network structures containing complex series and parallel combinations, the following is a simple example of determining the node weights:



(1) First assign the initial weight 1 to the nodes according to the series relations:

$\omega^{(1)} = 1$, $\omega^{(2))} = 1$, $\omega^{(3)} = 1$, $\omega^{(4)} = 1$, $\omega^{(5)} = 1$, $\omega^{(6)} = 1$, $\omega^{(7)} = 1$.

(2) Branch 1 (nodes 2 and 3) and Branch 2 (nodes 4, 5, 6) are in parallel; take the geometric mean of the two branches (the weights of the nodes inside each branch are divided by 2 on the original basis):

$\omega^{(1)} = 1$, $\omega^{(2))} = 0.5$, $\omega^{(3)} = 0.5$, $\omega^{(4)} = 0.5$, $\omega^{(5)} = 0.5$, $\omega^{(6)} = 0.5$, $\omega^{(7)} = 1$.

$\sum \omega^{(t)} = 4.5$, which lies between the geometric mean ($\sum \omega^{(t)} = 1$) and the product ($\sum \omega^{(t)} = 7$).

The above weights are derived only from the structural features of the network. In empirical studies, the final weights can also be determined by taking into account the importance of each node to the system.

2) The model with **lossy synergy (**$D^* < 0$)

$D^* < 0$ indicates lossy synergy in the network, and the node efficiency values are not unique:

$E^{(t)-} = (u^{(t)*} \cdot y_o^{(t)} + w^{(out)(t)*} \cdot z_o^{(out)(t)} - u_o^{(t)*}) / (v^{(t)*} \cdot x_o^{(t)} + w^{(in)(t)*} \cdot z_o^{(in)(t)}) \leq E^{(t)*}$,

If the dual model (9.25) is solved, then $E^{(t)-} = E^{(t)*} - \alpha^{(t)*} \leq E^{(t)*}$.

The system efficiency under lossy synergy is

$$E^- = \prod_{t=1}^{N} (E^{(t)-})^{\omega^{(t)}}, \ 1 \leq \sum \omega^{(t)} \leq N$$

The node efficiency values obtained under lossy synergy **may be not unique**.

3) For lossy synergy, the solutions are as follows

One solution is to jointly apply the weight ratio constraint model (also known as cone-ratio DEA, constrained cone DEA, assurance region DEA) to prevent the weights of one or more intermediates from being 0; with properly specified weight ratio constraints, lossless synergy can be achieved.

Another approach is the generalized network DEA based on full-dimensional efficient facets (FDEF) introduced later. In this model all weights are strictly greater than 0, and lossless synergy can be achieved by merely satisfying one simple condition (sufficient but not necessary): the network contains a node that only produces intermediates without consuming any (with $z^{(out)}$ and without $z^{(in)}$), and there is also a node that does not produce intermediates and only consumes them (with $z^{(in)}$ and without $z^{(out)}$). The two-stage network model satisfies this condition.

9.3.4 Compatibility with the Preceding Two-Stage Model

The two-stage network model is a special case of the generalized network model with a two-stage structure, and it satisfies the sufficient condition for **lossless synergy**; when the product form is used to compute the overall system efficiency, although the optimal solutions of the two models (the weights v, u, w) may differ, the resulting node efficiencies and system efficiency are the same.

9.4 Generalized Network DEA Based on Full-Dimensional Efficient Facets (FDEF)

A widely cited defect of classical DEA models is that efficiency measurement does not account for the input and output slacks that often appear in the models. For this reason, various non-radial models that measure efficiency directly based on slacks have been proposed. The input and output slack issue in the model is not a defect of classical DEA itself; rather, it arises because the data sample itself fails to establish a strongly efficient frontier covering the projection region of all samples, so the radial projection point falls on a weakly efficient frontier parallel to at least one coordinate axis. As shown in the figure below, the projection of DMU A is A_1, which lies on the weakly efficient frontier parallel to the vertical axis, and the slack of input x_2 is A_1B. The projection points of the various non-radial models lie on the strongly efficient frontier segment BC. Although non-radial models solve the problem that classical efficiency measurement ignores slacks, they introduce new problems: first, they weaken the economic meaning of the classical dual multiplier model; second, the projection points are often illogical—for example, the additive and SBM models determine the projection point by finding the "farthest distance", which is the least favorable to the DMU under evaluation. Taking the situation shown in the figure below as an example, in the additive and SBM models the projection point is C, which is farther from A, rather than the nearer B.

The slacks appearing in the classical envelopment model correspond to zero weights in the multiplier model. Another approach to addressing the slack problem in the classical model is the full-dimensional efficient facet (FDEF). In the example figure, segment BC of the frontier is an FDEF, and the radial projection based on the affine hull of BC (a straight line in two-dimensional coordinates, a hyperplane in multidimensional space) is A_2, a much more reasonable projection point compared to C.

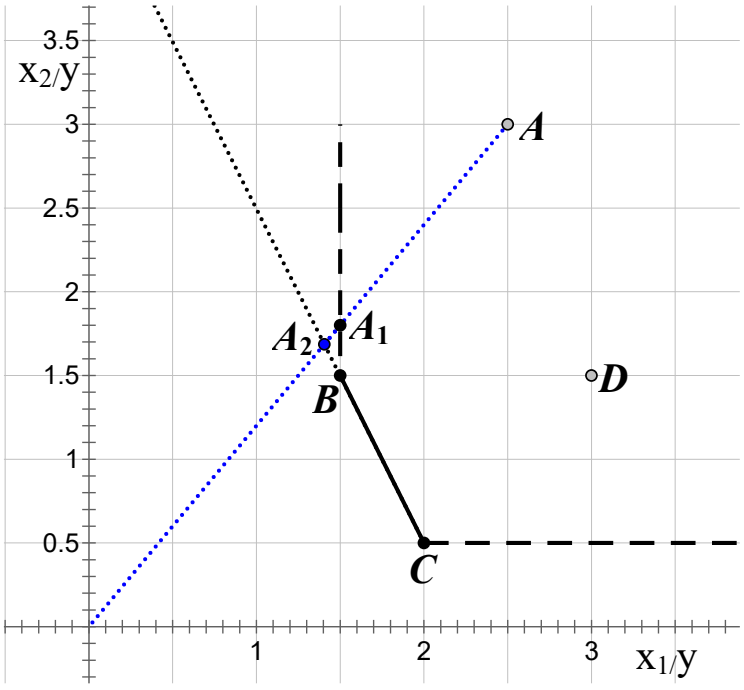


Figure 9-1 Illustration of projections (a CRS model with two inputs and one output)

This section presents a generalized network DEA model based on the affine hull of FDEFs, which eliminates the slack problems present in the classical model, including input and output slacks as well as

slacks of the intermediates. The elimination of intermediate slacks guarantees that the projected value of the intermediates as Stage 1 outputs exactly equals the projected value of the intermediates as Stage 2 inputs.

9.4.1 Introduction to Full-Dimensional Efficient Facets (FDEF)

In the convex-envelopment geometric theory of data envelopment analysis (DEA), full-dimensional efficient facets (Full-Dimensional Efficient Facets, FDEF) are one of the core concepts characterizing the frontier structure of the production possibility set (PPS).

1. Definition of Full-Dimensional Efficient Facets (FDEF)

In the DEA model containing m inputs and s outputs, the production possibility set PPS $\subset \mathbb{R}^{m+s}$ is a polyhedron. If an efficient facet formed where an efficient hyperplane on the boundary of this polyhedron intersects it has a geometric dimension exactly equal to $m+s-1$ (i.e., the highest possible degrees of freedom of the space), then the facet is called a **full-dimensional efficient facet (FDEF)** (Olesen & Petersen, 1996, 2003). From the multiplier (dual) perspective, all input multipliers v and output multipliers u corresponding to the normal vector of the hyperplane supporting the FDEF are strictly greater than zero (i.e., $v>0, u>0$).

2. Advantages of the DEA model based on the FDEF affine hull

DEA models constructed with FDEF-based constraints or identification have the following notable advantages:

Complete elimination of the zero-weight problem: conventional DEA models often exhibit the "zero weight" phenomenon due to multiplier degeneracy (i.e., some inputs or outputs are assigned zero weight), whereas FDEF ensures that all indicators play a role in the marginal evaluation.

Sound economic interpretation (marginal rates of substitution): since the normal vector of an FDEF is entirely positive, the marginal rates of substitution (MRS) derived from it are all finite, non-zero, and determinate, avoiding the economic breakdown on lower-dimensional efficient facets (LDEF), where MRS values are 0 or infinite.

Improved projection and target setting: projecting an inefficient decision-making unit (DMU) onto an FDEF hyperplane ensures that the improvement targets are strongly efficient, and free of slacks.

3. Methods for identifying all FDEFs in the sample of a DEA model

Methods for solving and identifying all FDEFs on the frontier of a polyhedron currently fall into two main categories:

Computational geometry algorithms based on the convex hull: using vertex enumeration and facet-generation algorithms from computational geometry, a convex hull is constructed over the set of DMU vertices, and the hyperplanes of dimension $m+s-1$ whose normal vectors are greater than 0 are extracted (Olesen & Petersen, 2003).

Iterative search methods based on mixed-integer programming (MIP): using duality theory, MIP is used to search for all hyperplanes with weights strictly greater than zero (Olesen & Petersen, 2003; Jahanshahloo, Lotfi, Rezai, & Balf, 2007; Jahanshahloo, Shirzadi, & Mirdehghan, 2009; Olyaie, Roshdi, Jahanshahloo, &

Asgharian, 2014).

4. Absence of FDEFs in the sample and severe frontier degeneracy

If a data set yields no FDEF at all when solved, this geometrically means that the envelopment frontier of the sample consists entirely of lower-dimensional efficient facets (LDEF). This carries a clear warning for both theory and application: it indicates severe frontier degeneracy, severely ill-conditioned data, or the curse of dimensionality (i.e., the sample size is severely insufficient relative to the dimensionality of the input-output indicators), causing the production frontier to “fold” or “collapse” into a lower-dimensional subspace in the high-dimensional space (Olesen & Petersen, 1996).

An effective remedy is to introduce a sufficient number of benchmark DMUs as the reference set of the model.

The existence of an FDEF in the sample does not mean that a satisfactory evaluation model can be built. If the model has only one or very few FDEFs, the DMU under evaluation may be projected onto a region far from the FDEF (lying on the supporting hyperplane of the FDEF but far away from the FDEF itself), yielding unrealistic target values. How to judge whether the FDEFs in a sample are sufficient is a question that still needs to be explored in both theory and practice. A practical way to judge the sufficiency of FDEFs is to check whether the projected values (targets) exceed a reasonable range.

9.4.2 Generalized Network DEA Model Based on the Affine Hull of FDEFs

A DEA model with an FDEF-extended frontier (Affine Hull of FDEFs) restricts the weights of the model to come from the set of FDEF hyperplanes. The generalized network DEA model with an FDEF-extended frontier is expressed as:

$$E^* = \max \Sigma((u^{(t)}\cdot y_o^{(t)} + w^{(out)(t)}\cdot z_o^{(out)(t)} - (\eta_2^{(t)} + \eta_4^{(t)})\cdot u_o^{(t)}) / (v^{(t)}\cdot x_o^{(t)} + w^{(in)(t)}\cdot z_o^{(in)(t)} + (\eta_1^{(t)} + \eta_3^{(t)})\cdot u_o^{(t)})) / N$$

$$\text{s.t. } \Sigma(w^{(in)(t)}\cdot z_o^{(in)(t)} - w^{(out)(t)}\cdot z_o^{(out)(t)} + (\eta_3^{(t)} + \eta_4^{(t)})\cdot u_o^{(t)}) = 0$$

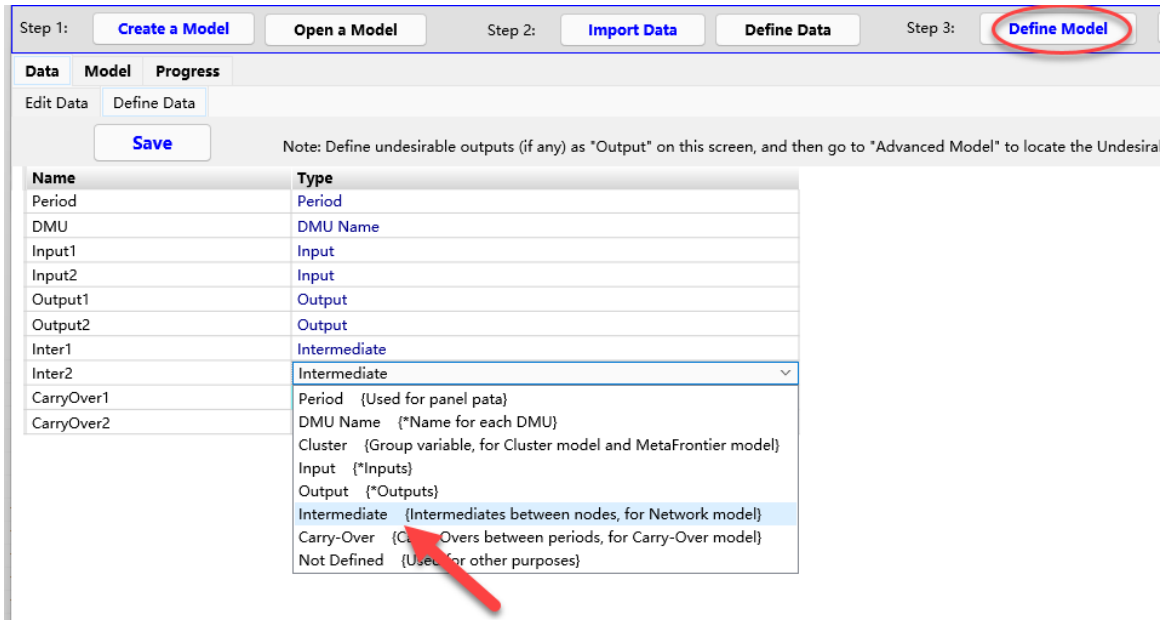
$$(v^{(t)}, u^{(t)}, w^{(in)(t)}, w^{(out)(t)}, u_o^{(t)}) \in F^{(t)} \qquad (9.26)$$

$F^{(t)}$ denotes the set of FDEF hyperplanes of stage t.

In the generalized network DEA model based on the affine hull of FDEFs, since all weights are strictly greater than 0, simply satisfying the following simple condition (sufficient but not necessary) achieves lossless synergy and unique decomposition of system efficiency: the network contains a node that only produces intermediates without consuming any (with $z^{(out)}$ and without $z^{(in)}$), and there is also a node that does not produce intermediates and only consumes them (with $z^{(in)}$ and without $z^{(out)}$).

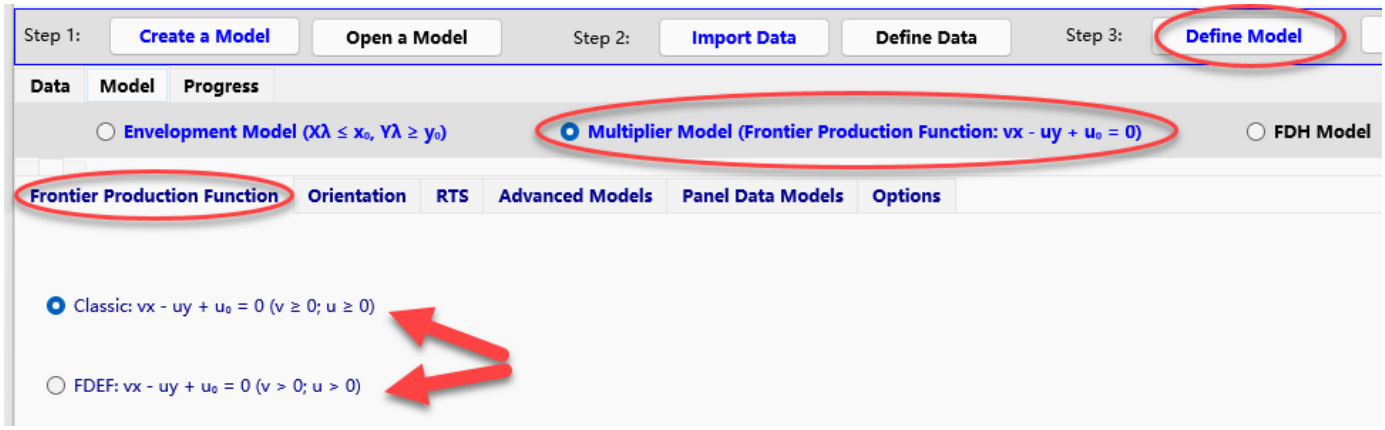
9.5 Software Demonstration of Network DEA

9.5.1 Defining the Data



9.5.2 Setting the General Options of the DEA Model

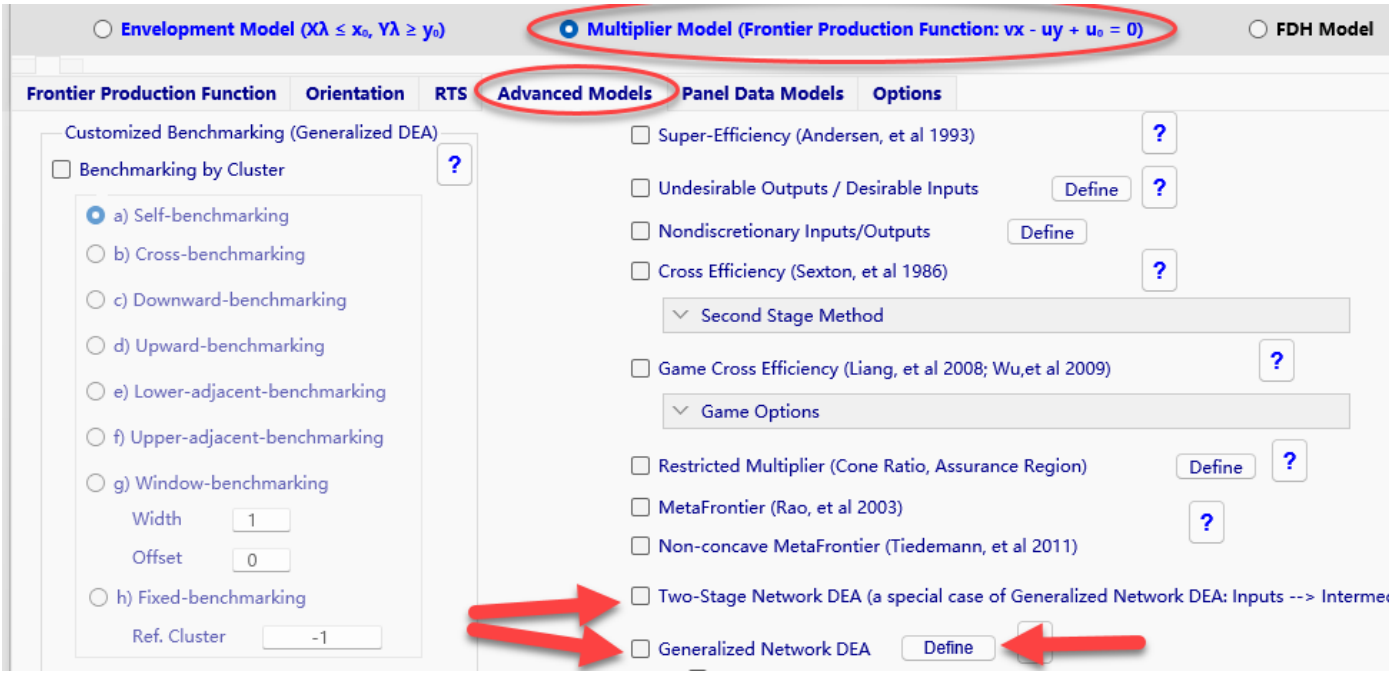
The production function type (classical frontier production function or FDEF affine-hull frontier production function), the Orientation, and the RTS.



9.5.3 Setting Other Model Options

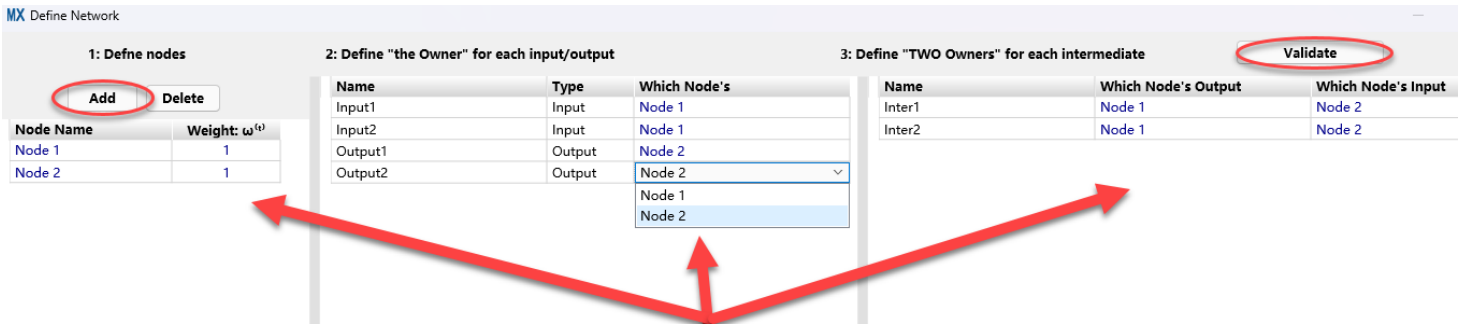
Select **Two-Stage Network DEA** or **Generalized Network DEA**.

Network DEA can be combined with all **panel data models**, including: Malmquist TFP index, HMB TFP index, window DEA, and Carry-Over DEA.



9.5.4 Defining the Network Structure

If "Generalized Network DEA" is selected, the network structure also needs to be defined. For each node of the generalized network model, the initial default weight $\omega^{(t)} = 1$, i.e., the system efficiency is the product of the node efficiencies; if the system efficiency is required to be the geometric mean of the node efficiencies, set the sum of the node weights equal to 1, i.e., $\Sigma\omega^{(t)} = 1$.



9.5.5 Results

The results include the overall system efficiency values, the weights of the inputs, outputs, and intermediates, the projected values (targets) of the inputs, outputs, and intermediates, the node efficiency values, and the benchmarks of the nodes and their coefficients (Lambda), etc.; the VRS models also include the returns-to-scale constant term ($u_0^{(t)}$), and the generalized network model also includes D* (used to determine whether the node efficiency values given by the network model are unique).

Appendix: Proof of the Multiplicative Decomposition of Efficiency in the CRS Two-Stage Network DEA

In the two-stage network DEA model, the overall network efficiency equals the product of the optimal stage efficiencies obtained from the Stage 1 and Stage 2 models:

$$\mathbf{E^* = E^{(1)*} \times E^{(2)*}}$$

1. **Proof of** $\boldsymbol{e^* \le e^{(1)*} \times e^{(2)*}}$

Let $(v', w^{(1)\prime}, w^{(2)\prime}, u')$ be an arbitrary **feasible solution of the overall system model**.

Using the intermediate constraint $w^{(1)\prime}z_0 = w^{(2)\prime}z_0$ of the network model, the objective function value corresponding to this feasible solution can be decomposed as:

$$\frac{u'y_0}{v'x_0} = \frac{w^{(1)\prime}z_0}{v'x_0} \cdot \frac{u'y_0}{w^{(2)\prime}z_0}$$

By the definition of feasible solutions:

1. The vector pair $(v', w^{(1)\prime})$ satisfies $\frac{w^{(1)\prime}z_j}{v\prime x_j} \le 1\ (\forall j)$, and therefore $(v', w^{(1)\prime})$ is a feasible solution of the Stage 1 model. By the definition of the maximum of $e^{(1)*}$:

$$\frac{w^{(1)\prime}z_0}{v'x_0} \le e^{(1)*}$$

2. The vector pair $(w^{(2)\prime}, u')$ satisfies $\frac{u\prime y_j}{w^{(2)\prime}z_j} \le 1\ (\forall j)$, and therefore $(w^{(2)\prime}, u')$ is a feasible solution of the Stage 2 model. By the definition of the maximum of $e^{(2)*}$:

$$\frac{u'y_0}{w^{(2)\prime}z_0} \le e^{(2)*}$$

Substituting the above two inequalities into the product decomposition yields:

$$\frac{u'y_0}{v'x_0} \le e^{(1)*} \times e^{(2)*}$$

Since the above inequality holds for **any** feasible solution in the feasible region of the network model, it also holds for the maximum over the feasible region (i.e., the optimal solution):

$$e^* \le e^{(1)*} \times e^{(2)*}$$

2. **Proof of** $\boldsymbol{e^* \ge e^{(1)*} \times e^{(2)*}}$

Let $(v^*, w^{(1)*})$ be an **optimal solution of the Stage 1 model**, satisfying:

$$e^{(1)*} = \frac{w^{(1)*}z_0}{v^*x_0}$$

Let $(w^{(2)*}, u^*)$ be an **optimal solution of the Stage 2 model**, satisfying:

$$e^{(2)*} = \frac{u^*y_0}{w^{(2)*}z_0}$$

Construct a positive scalar scaling factor:

$$\alpha = \frac{w^{(1)*}z_0}{w^{(2)*}z_0} > 0$$

Using the homogeneity of the Stage 2 fractional program (i.e., scaling the numerator and denominator weights proportionally does not change the constraints or the ratio), the Stage 2 optimal solution is linearly scaled to construct new weight vectors:

$$\tilde{w}^{(2)} = \alpha w^{(2)*}, \quad \tilde{u} = \alpha u^*$$

Verify the feasibility of the combined solution $(v^*, w^{(1)*}, \tilde{w}^{(2)}, \tilde{u})$ in the network model:

1. **The Stage 1 constraint**: $\frac{w^{(1)*}z_j}{v^*x_j} \le 1\ (\forall j)$ is clearly satisfied.
2. **The Stage 2 constraint**: for any j, we have:

$$\frac{\tilde{u}y_j}{\tilde{w}^{(2)}z_j} = \frac{\alpha u^* y_j}{\alpha w^{(2)*}z_j} = \frac{u^*y_j}{w^{(2)*}z_j} \le 1$$

so the constraints are satisfied.

3. **The intermediate flow equality constraint**:

$$\tilde{w}^{(2)}z_0 = \left(\alpha w^{(2)*}\right)z_0 = \left(\frac{w^{(1)*}z_0}{w^{(2)*}z_0}\right)\left(w^{(2)*}z_0\right) = w^{(1)*}z_0$$

The coupling constraint $w^{(1)}z_0 = w^{(2)}z_0$ is perfectly satisfied.

Therefore, $\left(v^*, w^{(1)*}, \tilde{w}^{(2)}, \tilde{u}\right)$ is a valid **feasible solution of the network model**.

Compute the objective function value of this feasible solution in the network model:

$$\frac{\tilde{u}y_0}{v^*x_0} = \frac{\alpha u^* y_0}{v^*x_0} = \frac{u^*y_0}{w^{(2)*}z_0} \cdot \frac{w^{(1)*}z_0}{v^*x_0} = e^{(2)*} \times e^{(1)*}$$

Since e^* is the global maximum of the network model, and the above combined solution is merely one particular solution in its feasible region, it must hold that:

$$e^* \geq e^{(1)*} \times e^{(2)*}$$

3. Conclusion

Combining the inequalities of the upper and lower bounds:

$$e^* \leq e^{(1)*} \times e^{(2)*} \text{ and } e^* \geq e^{(1)*} \times e^{(2)*}$$

By the antisymmetry of the order relation on real numbers, it necessarily follows that:

$$e^* = e^{(1)*} \times e^{(2)*}$$

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